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DIFFERENTIAL RINGS CONSTRUCTED FROM QUASI-PRIME IDEALS

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The purpose of this paper is to show that the basic results concerning commutative rings constructed from prime ideals in a given commutative ring via quotient rings and rings of fractions can be generalized to differential algebra. However, some care must be exercised, since it is not true, for example, that M is a maximal differential ideal in a differential ring A if and only if A/M is a differential field. The invalidity of this result is a consequence of the observation that some differential rings have maximal differential ideals which are not prime [1, p. 310]. Also, the radical of a differential ideal may fail to be a differential ideal [4, p. 12] so that for some differential rings A, the reduced ring A/N, where N is the nilradical of A, is not even a differential ring.

We begin by generalizing some earlier results for ordinary differential rings and obtaining some new ones for arbitrary differential rings. We then present the basic results for differential rings constructed via quotient rings and rings of fractions. The importance of these results is that they can play the same central role in differential algebra as do the corresponding results in commutative algebra. We conclude by giving some indication of the possible application of our results to problems in differential algebra.

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1. Preliminaries

Throughout the paper, A will denote a commutative ring with identity and $\Delta = \{\delta_1, \ldots, \delta_m\}$ a set of derivation operators on A, making A a differential ring. The free commutative semigroup generated by Δ will be denoted by Θ, so that Θ is the set of derivative operators on A. For any subset X of A, $r(X)$ will denote the

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radical of X, i.e. the intersection of all prime ideals in A containing X. Any unexplained notation or terminology will be standard, as in [5] or [8].

A subset X of A will be called differential if $\delta(X) \subset X$ for each $\delta \in \Delta$. For any subset X of A we define the differential of X to be the set $X_\Delta = \{x \in A \mid \theta x \in X$ for all $\theta \in \Theta\}$. Some of the properties of the operator $X \mapsto X_\Delta$ are given in the following.

Proposition 1.1. (i) *For any subset X of A, $X_\Delta \subset X$ and $X_{\Delta\Delta} = X_\Delta$.*

(ii) *For any subsets X and Y of A with $X \subset Y$, $X_\Delta \subset Y_\Delta$.*

(iii) *For any subset X of A, $X = X_\Delta$ if and only if X is differential.*

(iv) *For any family of subsets $\{X_i\}_{i \in I}$ of A,*

$$\left(\bigcap_{i \in I} X_i\right)_\Delta = \bigcap_{i \in I} (X_i)_\Delta \quad \textit{and} \quad \bigcup_{i \in I} (X_i)_\Delta \subset \left(\bigcup_{i \in I} X_i\right)_\Delta.$$

(v) *If B is any differential ring and $f: A \to B$ is any differential ring homomorphism, then for any subsets X of A and Y of B, $f^{-1}(Y_\Delta) = f^{-1}(Y)_\Delta$ and $f(X_\Delta) \subset f(X)_\Delta$ with equality if f is injective.*

Proof. Immediate from the definition.

It follows that for any subset X of A, X_Δ is the largest differential subset of A contained in X. Furthermore, the collection of differential subsets of A is a complete lattice, and a differential ring homomorphism induces two lattice homomorphisms via inverse and direct image.

Before we present a lemma which will be useful in proving some of the algebraic properties of the operator $X \to X_\Delta$, we observe that Θ has the following inductive property. If S is a subset of Θ such that

(1) $1 \in S$; and

(2) for all $\theta \in \Theta$, if $\theta' \in S$ for all $\theta' \in \Theta$ with $\theta' \mid \theta$ (i.e. θ' dividing θ in Θ) and $\theta' \neq \theta$, then $\theta \in S$;

then $S = \Theta$.

For any subset X of A, let $U(X) = \{x \in X \mid$ there exists $y \in X$ with $xy = 1\}$. Thus if B is a subring of A, $U(B)$ is the set of invertible elements in B.

Lemma 1.2. *If B is a subring of A, then $U(B_\Delta) = U(B) \cap B_\Delta$.*

Proof. Clearly $U(B_\Delta) \subset U(B) \cap B_\Delta$, so let $x \in B$ be such that $\theta x \in B$ for all $\theta \in \Theta$ and suppose that $xy = 1$ for some $y \in B$. We need to show that $\theta y \in B$ for each $\theta \in \Theta$. We may assume that $\theta \neq 1$ and that for each $\theta' \in \Theta$ with $\theta' \mid \theta$ and $\theta' \neq \theta$, $\theta' y \in B$. It follows that $0 = \theta(xy) = x\theta y + \sum \binom{\theta}{\theta'} \theta' x \theta'' y$, the sum taken over all pairs (θ', θ'') in $\Theta \times \Theta$ with $\theta = \theta'\theta''$ and $\theta' \neq 1$, as in [8, p. 60]. From this it follows that $\theta y = -y(\sum \binom{\theta}{\theta'} \theta' x \theta'' y)$ and hence $\theta y \in B$.

Proposition 1.3. *Let X be a subset of A. If X is any of the following, the same is true of X_Δ.*

(i) *A subring of A.*
(ii) *An ideal of A.*
(iii) *A semi-local subring of A (i.e. a subring having finitely many maximal ideals).*
(iv) *A local subring of A.*
(v) *A subfield of A.*

Proof. This follows from Lemma 1.2 and from Proposition 1.4 [6, p. 241].

Proposition 1.4. *Suppose that A has characteristic $n>0$, and let P be a prime ideal in A. Then $r(P_\Delta)=P$.*

Proof. Since P is a prime ideal and $P_\Delta \subset P$ by Proposition 1.1(i), we see that $r(P_\Delta)\subset P$. However, for any $a\in P$, one checks easily that $a^n \in P_\Delta$.

Proposition 1.5. *Suppose that A has characteristic zero, let S be the multiplicative subset of A consisting of all $n\cdot 1$ where n is a positive integer, and let P be a prime ideal in A.*
(i) *If $P\cap S\neq\emptyset$, then $r(P_\Delta)=P$.*
(ii) *If $P\cap S=\emptyset$, then P_Δ is a prime ideal in A.*

Proof. (i) If $P\cap S\neq\emptyset$, there is a prime integer p such that $p\cdot 1\in P$, and so $a^p\in P_\Delta$ for any $a\in P$, showing $P\subset r(P_\Delta)$. But since $r(P_\Delta)\subset P$ for any prime ideal P, the result follows.

(ii) First observe that since Θ is the free commutative semigroup generated by $\Delta=\{\delta_1,\dots,\delta_m\}$, we have that $\Theta\cong \mathbf{N}^m$ where $\mathbf{N}$ denotes the natural numbers. It follows that there is an injection $\Theta\to\mathbf{N}\times\mathbf{N}^m$ induced by the mapping $\operatorname{ord}:\Theta\to\mathbf{N}$, where for any $\theta=\prod_{\delta\in\Delta}\delta^{e(\delta)}\in\Theta$, $\operatorname{ord}\theta=\sum_{\delta\in\Delta}e(\delta)$. The lexicographical order on $\mathbf{N}\times\mathbf{N}^m$ then induces a total ordering on Θ. Now suppose that $a,b\in A$ are such that $a\notin P_\Delta$ and $b\notin P_\Delta$. Then there exist $\theta'\in\Theta$ and $\theta''\in\Theta$ such that $\theta'a\notin P$, $\theta''b\notin P$, $\theta a\in P$ for all $\theta<\theta'$ and $\theta b\in P$ for all $\theta<\theta''$. Now consider

$$\theta'\theta''(ab)=\binom{\theta'\theta''}{\theta'}\theta'a\theta''b+\sum\binom{\theta'\theta''}{\theta_1}\theta_1a\theta_2b,$$

the sum taken over all pairs (θ_1,θ_2) in $\Theta\times\Theta$ with $\theta_1\theta_2=\theta'\theta''$ and $\theta_1\neq\theta'$. But for any such pair (θ_1,θ_2), either $\theta_1<\theta'$ or $\theta_1>\theta'$, in which case $\theta_2<\theta''$. It follows that

$$\sum\binom{\theta'\theta''}{\theta_1}\theta_1a\theta_2b\in P,$$

where $\theta_1\neq\theta'$, and since $P\cap S=\emptyset$, $\binom{\theta'\theta''}{\theta'}\theta'a\theta''b\notin P$, so that $ab\notin P_\Delta$, showing P_Δ is a prime ideal in A.

Corollary 1.6. *Suppose that A contains the rational numbers $\mathbf{Q}$. If P is a prime ideal in A, so is P_Δ. Similarly if I is a radical ideal in A, so is I_Δ.*

Proof. If $\mathbf{Q} \subset A$, then for any positive integer n, $n \cdot 1$ is invertible in A, so that if P is a prime ideal in A, P_Δ is prime by Proposition 1.5. If I is a radical ideal in A, I is an intersection of prime ideals, so that I_Δ is a radical ideal as well by Proposition 1.1(iv).

Lemma 1.7. *Let P be a prime ideal in A and let S be a multiplicative subset of A such that $P \cap S = \emptyset$. Then in the differential ring $S^{-1}A$ we have $(S^{-1}P)_\Delta = S^{-1}P_\Delta$.*

Proof. Let $a/s \in S^{-1}P_\Delta$; we will show that $\theta(a/s) \in S^{-1}P$ for all $\theta \in \Theta$. For $\theta = 1$ this is immediate, so suppose that $\theta'(a/s) \in S^{-1}P$ for all $\theta' \in \Theta$, $\theta' \mid \theta$, $\theta' \neq \theta$. Since $(a/s) \cdot (s/1) = a/1$ in $S^{-1}A$, we have

$$\theta(a/s) = \left[\theta(a/1) - \sum \binom{\theta}{\theta'} \theta'(a/s) \cdot \theta''(s/1)\right](1/s), \qquad (*)$$

the sum taken over all pairs (θ', θ'') with $\theta = \theta'\theta''$ and $\theta' \neq \theta$. Since $\theta(x/1) = \theta x/1$ for all $x \in A$ and $\theta \in \Theta$, we see by $(*)$ that $\theta(a/s) \in S^{-1}P$. On the other hand, if we assume that $\theta(a/s) \in S^{-1}P$ for each $\theta \in \Theta$, then again by $(*)$ we see that $\theta a/1 \in S^{-1}P$. Since P is prime and disjoint from S, it follows that $\theta a \in P$, so that $a/s \in S^{-1}P_\Delta$.

As in [6], we say that A is special if for each prime ideal P in A, P_Δ is also a prime ideal in A. It follows from Corollary 1.6 that any differential ring containing the rational numbers $\mathbf{Q}$ is special. We record the following propositions concerning special differential rings.

Proposition 1.8. *The following are equivalent.*

(1) *A is special.*

(2) *Every minimal prime divisor of a differential ideal in A is a differential ideal in A.*

(3) *The radical of a differential ideal in A is a differential ideal in A.*

(4) *If I is a differential ideal in A and S a multiplicative subset of A disjoint from I, then every ideal in A maximal among differential ideals in A which contain I and are disjoint from S is a prime ideal in A.*

(5) *If I is a radical ideal in A, then I_Δ is also a radical ideal in A.*

(6) *If $S = \{n \cdot 1 \mid n > 0\}$ and P is a prime ideal in A such that $P \cap S \neq \emptyset$, then $P_\Delta = P$.*

Proof. Follows from Proposition 2.1 [7, p. 380].

Proposition 1.9. (i) *If A is special and I is any differential ideal in A, then A/I is special.*

(ii) *If A is special and S is any multiplicative subset of A, then $S^{-1}A$ is special.*

(iii) *A is special if and only if A_p is special for each prime ideal P in A.*

(iv) *If $A_1, \ldots, A_n$ is any finite family of differential rings and $A = A_1 \times \cdots \times A_n$, then A is special if and only if each A_i is special.*

Proof. Follows from Propositions 1.6, 1.8, 1.10 and Corollary 1.9 [6, pp. 242–243].

Recall from [7] that an ideal Q in A is called quasi-prime if there is a multiplicative subset S of A such that Q is maximal among differential ideals in A disjoint from S. It is clear from [5, Prop. 1, p. 1] that the notion of a quasi-prime ideal in a differential ring is a generalization of the notion of a prime ideal in a commutative ring. Moreover, it is a better generalization than the usual notion of a prime differential ideal for several reasons. First, every prime differential ideal is a quasi-prime ideal. Moreover, every maximal differential ideal is a quasi-prime ideal (but is not necessarily a prime differential ideal as we have noted above). Furthermore, in many of the cases usually considered in differential algebra, such as when the differential ring contains the rational numbers, quasi-prime ideals are the same as prime differential ideals. Finally, prime ideals and quasi-prime ideals are closely related in many ways, as the following shows.

Proposition 1.10. *Let Q be an ideal in A. The following are equivalent.*

(1) *Q is quasi-prime.*
(2) *Q is primary and $Q=r(Q)_\Delta$.*
(3) *$r(Q)$ is prime and $Q=r(Q)_\Delta$.*
(4) *There is a prime ideal P in A such that $Q=P_\Delta$.*

Proof. Follows from Proposition 2.2 [7, p. 380].

Denote by $\mathrm{Spec}_\Delta A$ and $\mathrm{Quas}\, A$ the sets of prime differential ideals and quasi-prime ideals in A respectively. We have the following.

Corollary 1.11. *There is a natural surjection $\Delta : \mathrm{Spec}\, A \to \mathrm{Quas}\, A$ and a natural injection $r : \mathrm{Quas}\, A \to \mathrm{Spec}\, A$ such that $\Delta \cdot r(Q) = Q$ for all $Q \in \mathrm{Quas}\, A$. If the characteristic of A is positive, or if the characteristic of A is zero and each nonzero prime ideal in A contains an element $p \cdot 1$ where p is a prime integer, then Δ (and hence r) is a bijection. If A contain the rational numbers, then* $\mathrm{Quas}\, A = \mathrm{Spec}_\Delta A$.

Proof. The existence of Δ and r follows immediately from Proposition 1.10. The naturality of Δ and r means that for any differential ring homomorphism $f: A \to B$ there are commuting diagrams of mappings

$$\begin{array}{ccc} \mathrm{Spec}\, B & \xrightarrow{f^{-1}} & \mathrm{Spec}\, A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ \mathrm{Quas}\, B & \xrightarrow{f^{-1}} & \mathrm{Quas}\, A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Quas}\, B & \xrightarrow{f^{-1}} & \mathrm{Quas}\, A \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ \mathrm{Spec}\, B & \xrightarrow{f^{-1}} & \mathrm{Spec}\, A \end{array}$$

i.e. for any prime ideal P in B, $f^{-1}(P)_\Delta = f^{-1}(P_\Delta)$ and for any quasi-prime ideal Q in

B, $f^{-1}(r(Q))=r(f^{-1}(Q))$. The first follows from Proposition 1.1(v) and the second is immediate. The remainder follows from Propositions 1.4 and 1.5 and Corollary 1.6.

The following result is immediate from Propositions 2.6 and 2.7 [7, pp. 382–383].

Proposition 1.12. (i) *If I is a differential ideal in A, there is a one-to-one correspondence between quasi-prime ideals in A/I and quasi-prime ideals in A which contain I.*

(ii) *If S is a multiplicative subset of A, there is a one-to-one correspondence between quasi-prime ideals in $S^{-1}A$ and quasi-prime ideals in A disjoint from S.*

Lemma 1.13. *Let $(y_1, \ldots, y_n)$ be a finite family of differential indeterminates over A and let B denote the differential polynomial ring $A\{y_1, \ldots, y_n\}$. Then for any ideal I in A, $(IB)_\Delta = I_\Delta B$.*

Proof. We first claim that the set W of differential monomials in B can be totally ordered so that for any $\theta \in \Theta$, $\theta \neq 1$, any $M \in W$ and any term aN in θM, where $N \in W$ and $0 \neq a \in A$, $M < N$. To see this, first observe that the set D of all derivatives θy_j, where $\theta \in \Theta$, $j=1, \ldots n$, can be totally ordered so that for any $u \in D$ and any $\theta \in \Theta$, $u \leq \theta u$ by [8, Lemma 15, p. 49]. Then any $M \in W$ can be written as a finite product of factors of the form u^m, where $u \in D$ and $m>0$, and the total ordering of D can be extended to a total ordering of W as follows. First, we will write any such monomial $M \in W$ as $M=\prod_{i=1}^{k}(u_i)^{m_i}$ where $u_1 > \cdots > u_k$ and $m_i>0$ for $i=1, \ldots, k$. Then if $M' \in W$, say $M'=\prod_{i=1}^{k'}(u_i')^{m_i'}$ where $u_1' > \cdots > u_{k'}'$ and $m_i'>0$ for $i=1, \ldots, k'$, we will say that M' has higher rank than M if there is a positive integer j such that for all $i<j$, $u_i=u_i'$ and $m_i=m_i'$, and either $u_j<u_j'$, or $u_j=u_j'$ and $m_j<m_j'$. Then it is immediate that for any $\theta \in \Theta$, $\theta \neq 1$, and any $M \in W$, the monomial of each term of θM has higher rank than M.

It is clear that $I_\Delta B \subset (IB)_\Delta$. If $F \in B - I_\Delta B$, then we can assume that $F=aM+G$, where $M \in W$, $a \in A - I_\Delta$, $G \in B$, and the monomial of each term of G has higher rank than M (relative to the total order defined above). Since $a \notin I_\Delta$, there is some $\theta \in \Theta$ with $\theta a \notin I$. Therefore $\theta F=(\theta a)M+\sum \binom{\theta}{\theta'}\theta' a \theta'' M+\theta G$, the sum taken over all pairs (θ', θ'') with $\theta=\theta'\theta''$ and $\theta'' \neq 1$. But the monomial of each term of $\theta'' M$ has higher rank than M, and similarly the monomial of each term of θG has higher rank than M. Therefore $\theta F \notin IB$, and hence $I_\Delta B=(IB)_\Delta$.

Proposition 1.14. *Let $(y_1, \ldots, y_n)$ be a finite family of differential indeterminates over A and let B denote the differential polynomial ring $A\{y_1, \ldots, y_n\}$. Then Q is a quasi-prime ideal in A if and only if QB is a quasi-prime ideal in B.*

Proof. One direction is clear, since $QB \cap A = Q$, so suppose that Q is a quasi-prime ideal in A, let $P=r(Q)$, and suppose that $F, G \in B$ are such that $FG \in QB$ and $F \notin QB$. Then there is a finite subset X of the set of derivatives θy_j, where $\theta \in \Theta$, $j=1, \ldots, n$, such that $FG \in QA[X]$ and $F \notin QA[X]$. It follows from Proposition 1.10 and [3, Exercise 15, p. 293] that $QA[X]$ is primary and $PA[X]$ is its radical, so that $G^n \in QA[X]$ for some $n > 0$ and hence $G \in PA[X]$. Therefore QB is primary and $r(QB) = PB$. Since $P_\Delta = Q$, it follows from Lemma 1.13 that $(r(QB))_\Delta = QB$, so that by Proposition 1.10, QB is a quasi-prime ideal in B.

Since the notion of a quasi-prime ideal in a differential ring is a better generalization to differential algebra of the notion of a prime ideal in a commutative ring than that of a prime differential ideal, then one must consider the types of differential rings which arise when one constructs rings of fractions and quotient rings using quasi-prime ideals. In the next section we see that these differential rings are reasonably well-behaved.

2. Differential quotient rings and rings of fractions

We begin by generalizing suitably the notion of a differential domain to the case at hand. We will say that A is a quasi-domain if every zero-divisor in A is nilpotent and every non-zero element has some derivative (perhaps of order zero) that is not nilpotent. It is immediate from the definition that every differential domain is a quasi-domain. The following is an immediate consequence of Proposition 1.10.

Proposition 2.1. *The following are equivalent.*

(1) *A is a quasi-domain.*

(2) *$\{0\}$ is a quasi-prime ideal in A.*

(3) *The nilradical N of A is a prime ideal and there are no proper differential ideals contained in N.*

The basic results in commutative algebra concerning integral domains generalize as follows.

Proposition 2.2. *Q is a quasi-prime ideal in A if and only if A/Q is a quasi-domain.*

Proof. Immediate from Propositions 2.1 and 1.12(i).

Proposition 2.3. *If A is a quasi-domain and S a multiplicative subset of A, then $S^{-1}A$ is a quasi-domain.*

Proof. Immediate from Propositions 2.1 and 1.12(ii).

Proposition 2.4. *Let $(y_1, \ldots, y_n)$ be a finite family of differential indeterminates over A. Then $A\{y_1, \ldots, y_n\}$ is a quasi-domain if and only if A is a quasi-domain.*

Proof. This is immediate from Propositions 2.1 and 1.14.

We will say that a differential ideal M in A is quasi-maximal if $r(M)$ is a maximal ideal in A and $M=r(M)_\Delta$. It is clear that any quasi-maximal ideal is a maximal differential ideal (and therefore a quasi-prime ideal), but not every maximal differential ideal is quasi-maximal. For example, in the differential ring $\mathbf{Q}[t]$ where t is an indeterminate over $\mathbf{Q}$, $\Delta=\{\delta\}$ and $\delta t=1$, $\{0\}$ is a maximal differential ideal but is not quasi-maximal.

To fix terminology, we will call any (not necessarily Noetherian) ring having a unique maximal ideal a local ring. (Such a ring is sometimes called a quasi-local ring by some authors who reserve the term 'local' for the Noetherian case.)

We will say that A is a q-local ring if A is a local ring whose maximal ideal M satisfies $M=r(M_\Delta)$ (or equivalently is such that M_Δ is quasi-maximal). Any local differential ring (i.e. a local ring whose maximal ideal is differential) is q-local.

Proposition 2.5. *Let Q be a quasi-prime ideal in A and let $S=A-r(Q)$. Then $S^{-1}A$ is q-local.*

Proof. Since $r(Q)=P$ is a prime ideal in A, $S^{-1}A$ is local. Furthermore, the maximal ideal $S^{-1}P$ in $S^{-1}A$ satisfies $r((S^{-1}P)_\Delta)=r(S^{-1}P_\Delta)=S^{-1}r(P_\Delta)=S^{-1}P$, where the first equality follows from Lemma 1.7, and the second is immediate, so that $S^{-1}A$ is q-local.

If Q is a quasi-prime ideal in A and $S=A-r(Q)$, we will denote by A_Q the q-local ring $S^{-1}A$, and A_Q will be called the q-local ring of A at Q.

Proposition 2.6. *Let U denote the multiplicative set of invertible elements of A. Then A is q-local if and only if A satisfies the conditions:*

(1) *if $x,y\in A$ are such that $x+y\in U$, then either $x\in U$ or $y\in U$, and*

(2) *if $x\in A$ is such that for every $n>0$ there exists $\theta\in\Theta$ such that $\theta(x^n)\in U$, then $x\in U$.*

Proof. It is clear that (1) is equivalent to A being local, and it is not difficult to check that (2) is equivalent to the condition $M\subset r(M_\Delta)$ where $M=A-U$ is the maximal ideal of A.

Proposition 2.7. *Let S be a saturated multiplicative subset of A (i.e. S satisfies $xy\in S$ if and only if $x\in S$ and $y\in S$). Then $S^{-1}A$ is q-local if and only if S is the complement in A of the radical of a quasi-prime ideal in A.*

Proof. Suppose that $S^{-1}A$ is q-local. Let M denote the maximal ideal in $S^{-1}A$, so that $M=S^{-1}P$ for some prime ideal P in A disjoint from S. Therefore $S^{-1}P=r((S^{-1}P)_\Delta)=r(S^{-1}P_\Delta)=S^{-1}r(P_\Delta)$, and since S is saturated, $S=A-P=A-r(P_\Delta)$. The converse is just Proposition 2.5.

We say that A is a quasi-field if every non-unit in A is nilpotent and every non-zero element has some derivative (perhaps of order zero) that is not nilpotent. It is clear that every differential field is a quasi-field, and that every quasi-field is a quasi-domain. Furthermore, every quasi-field is q-local, since the nilradical is then the maximal ideal. Note also that the subring of constants of a quasi-field is a quasi-field (actually, a field).

Proposition 2.8. *Let U denote the set of invertible elements in A and N the nilradical of A. The following are equivalent.*

(1) *A is a quasi-field.*

(2) *$\{0\}$ is a quasi-maximal ideal in A.*

(3) *A satisfies the conditions:*

  (i) *For any $x \in A$, either $x \in U$ or $x \in N$.*

  (ii) *If x is such that for every $\theta \in \Theta$ there exists $n > 0$ such that $(\theta x)^n = 0$, then $x = 0$.*

(4) *A has a unique prime ideal and no proper differential ideals.*

Proof. Immediate.

Proposition 2.9. *M is a quasi-maximal ideal in A if and only if A/M is a quasi-field.*

Proof. Suppose first that M is a quasi-maximal ideal in A and that $a + M$ is not nilpotent in A/M. Since $r(M)$ is a maximal ideal in A, there exists $x \in A$ with $1 - ax \in r(M)$, and hence $(1 - ax)^n \in M$ for some $n > 0$. Expanding $(1 - ax)^n$, it follows that $1 - ay \in M$ for some $y \in A$, so that $a + M$ is a unit in A/M. Also, if $a + M$ is any non-zero element in A/M, then since $M = r(M)_\Delta$, we see that for some $\theta \in \Theta$, $\theta(a + M) = \theta a + M$ is not nilpotent, so that A/M is a quasi-field. Conversely, suppose that A/M is a quasi-field and that $a \in A - r(M)$. Then $a + M$ is not nilpotent in A/M, so that there is an $x \in A$ with $1 - ax \in M$. It follows that $r(M)$ is a maximal ideal in A. Now if $a \in A - M$, then $a + M$ has some derivative that is not nilpotent in A/M, so that $a \in A - r(M)_\Delta$. Hence M is a quasi-maximal ideal in A.

If A is q-local and M the unique quasi-maximal ideal in A, then A/M will be called the differential residue quasi-field of A.

Proposition 2.10. *If A is a quasi-domain and S the multiplicative set of non-nilpotent elements in A, then $S^{-1}A$ is a quasi-field.*

Proof. It is clear that every non-unit in $S^{-1}A$ is nilpotent. Moreover, if $a/s \in S^{-1}A$ is such that for every $\theta \in \Theta$, $\theta(a/s)$ is nilpotent, then since the nilradical of A is a prime ideal and since

$$\theta(a/s) = \left(\theta(a/1) - \sum \binom{\theta}{\theta'} \theta'(a/s)\theta''(s/1)\right)(1/s),$$

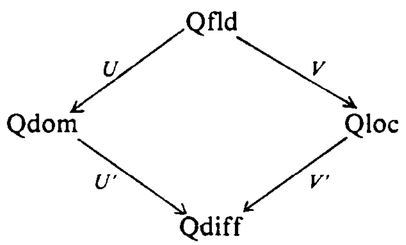
the sum taken over pairs (θ', θ'') with $\theta = \theta'\theta''$ and $\theta' \neq \theta$, it follows that $\theta a/s$ is nilpotent in $S^{-1}A$, and hence θa is nilpotent. But every non-zero element in A has some derivative which is not nilpotent, so that $a = 0$ and hence $a/s = 0$.

If A is a quasi-domain and S the multiplicative set of non-nilpotent elements of A, we call $S^{-1}A$ the differential quasi-field of quotients of A. We note that the canonical differential ring homomorphism $h : A \to S^{-1}A$ is a monomorphism since any zero-divisor in A is nilpotent.

Proposition 2.11. *Let A be a quasi-domain and S the set of non-nilpotent elements of A. If F is any differential quasi-field and $f : A \to F$ any differential ring monomorphism, there is a unique differential ring monomorphism $\bar{f} : S^{-1}A \to F$ such that $f = \bar{f} \cdot h$.*

Proof. This is immediate, since if s is any non-nipotent element in A, $f(s)$ is a unit in F. For otherwise, $f(s)$ would be nilpotent in F, and hence s would be nilpotent in A.

The preceding results can be interpreted very nicely from a categorical viewpoint. To this end, let Qfld denote the category of differential quasi-fields, where the morphisms are differential monomorphisms; Qloc the category of differential q-local rings, where the morphisms are differential local homomorphisms; Qdom the category of differential quasi-domains, where the morphisms are differential monomorphisms; and Qdiff the category whose objects are pairs (A, Q) where A is a differential ring and Q a quasi-prime ideal in A, where the morphisms $f : (A, Q) \to (A', Q')$ are differential homomorphisms $f : A \to A'$ with $Q = f^{-1}(Q')$. Then consider the diagram of functors



where U and V are inclusions, and U' and V' are defined objectwise by $U'A = (A, \{0\})$ and $V'A = (A, M)$ where M is the quasi-maximal ideal of A. It is clear that $U'U = V'V$, and moreover, each of the functors U, U', V, and V' has a left adjoint. In particular, that U' has a left adjoint follows from Proposition 2.2, that V' has a left adjoint follows from Proposition 2.5, that V has a left adjoint follows from Proposition 2.9, and that U has a left adjoint follows from Propositions 2.10 and 2.11. The composite of the left adjoints to U and U', and the composite of the left adjoints to V and V' both provide left adjoints for the functor $U'U = V'V$ by

[9, Theorem 1, p. 101], and therefore they are naturally isomorphic by [9, Corollary 1, p. 83]. Hence we have proved the following.

Theorem 2.12. *Let A be a differential ring and Q a quasi-prime ideal in A. Then the differential quasi-field of quotients of A/Q is naturally isomorphic to the differential residue quasi-field of A_Q.*

As we have seen, differential rings constructed from a given differential ring by rings of fractions and quotient rings using quasi-prime ideals behave just as their algebraic counterparts, and include a greater variety of differential rings than one obtains by restricting only to prime differential ideals.

The techniques developed in this paper can be applied to other problems in differential algebra as well. For example, a very useful technique in commutative algebra involves the selection of a maximal element from a given partially ordered set, such as the set of ideals of a given commutative ring which are disjoint from a multiplicative subset of the ring, the set of local subrings of a given field or the set of homomorphisms from subrings of a given field. As we have pointed out, in differential algebra, differential ideals which are maximal may fail to be prime (hence quasi-prime ideals arise). Similarly, maximal elements in other partially ordered sets in differential algebra do not behave exactly as their algebraic counterparts (cf., e.g., [2] or [10]). It seems quite likely that our results can be applied to these other situations as well, but to do so would take us beyond the scope of the present paper.

References

[1] P. Blum, Complete models of differential fields, Trans. Amer. Math. Soc. 137 (1969) 309–325.

[2] P. Blum, Extending differential specializations, Proc. Amer. Math. Soc. 24 (1970) 471–474.

[3] N. Bourbaki, Elements of Mathematics, Commutative Algebra (Addison–Wesley, Reading, MA, 1972).

[4] I. Kaplansky, An Introduction to Differential Algebra (Hermann, Paris, 1957).

[5] I. Kaplansky, Commutative Rings (University of Chicago Press, Chicago, 1974).

[6] W. Keigher, Prime differential ideals in differential rings, in: H. Bass, P.J. Cassidy and J. Kovacic, Eds., Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin (Academic Press, New York, 1977), 239–249.

[7] W. Keigher, Quasi-prime ideals in differential rings, Houston J. Math. 4 (1978) 379–388.

[8] E. Kolchin, Differential Algebra and Algebraic Groups (Academic Press, New York, 1973).

[9] S. MacLane, Categories for the Working Mathematician (Springer, New York, 1971).

[10] S. Morrison, Differential specializations in integral and algebraic extensions, Amer. J. Math. 101 (1979) 1381–1399.